Pre-class Warm-up!!!

1. $\frac{d}{dx} \sin 2x^2 =$

a. $\cos 4x$

b. $2x^2 \sin 2x^2$

c. $4x \cos 2x^2$ ✓

d. $2x^2 \cos 4x$

e. None of the above

2. To do question 1, we use:

a. a calculator

b. Leibniz′ rule

c. the chain rule ✓

d. guess work

e. None of the above

2.5 The chain rule and other properties of the derivative.

Mostly this section is about the chain rule, and the most important thing is what it looks like, why you might expect it to be this way, and how to do the HW questions

$u = u(x) \qquad y = y(u)$

$$\left.\frac{dy}{dx}\right|_a = \left.\frac{dy}{du}\right|_{u(a)} \cdot \left.\frac{du}{dx}\right|_a$$

Review of the 1-variable case.

Informally the chain rule says
dy/dx = (dy/du)(du/dx)
when y is a function of u and u a function of x.

E.g. u = x^2 + x, y = 2u^2

- y = 2(x^2+x)^2 = 2x^4 + 4x^3 + 2x^2
- dy/dx = 8x^3 + 12x^2 + 4x

- dy/du = 4u   du/dx = 2x+1
- dy/dx = (4u) (2x+1) = 4(x^2+x) (2x+1)
  = 8x^3 + 12x^2 + 4x

multiplication

Or: $(f \circ g)'(a) = f'(g(a))\, g'(a)$

Special case of the chain rule

Let c : R -> R^3 and f : R^3 -> R be

(x, y, z)

c(t) = (t^2, 2t, sin t), f(x,y,z) = 2x + yz^2

$f \circ c : \mathbb{R} \to \mathbb{R}$

Find df/dt

Question: was it right to use d rather than ∂ just now?

a. Yes ✓
b. No

It's also correct to write ∂ always.

$$Df(a) = Df(c(a)) \circ Dc(a)$$

matrix multiplication

The chain rule says

$$\left.\frac{df}{dt}\right|_a = \left.\frac{\partial f}{\partial x}\right|_{c(a)} \left.\frac{\partial x}{\partial t}\right|_a + \left.\frac{\partial f}{\partial y}\right|_{c(a)} \left.\frac{\partial y}{\partial t}\right|_a + \left.\frac{\partial f}{\partial z}\right|_{c(a)} \left.\frac{\partial z}{\partial t}\right|_a$$

We should really evaluate these derivatives appropriately.

$$\frac{df}{dt} = 2 \cdot 2t + z^2 \cdot 2 + y \cos t$$

$$= 4t + 2\sin^2 t + 2t \cos t.$$

Notice: the derivative matrices of f and c are

$$Df = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right], \quad Dc = \begin{bmatrix} \frac{\partial c_1}{\partial t} \\ \frac{\partial c_2}{\partial t} \\ \frac{\partial c_3}{\partial t} \end{bmatrix}$$

and $\frac{df}{dt} = Df \cdot Dc$ matrix multiplication.

The chain rule

Theorem 11. Let $g : R^n \to R^m$ and $f : R^m \to R^p$ be differentiable. Then $f \circ g$ is differentiable and
$D(f \circ g)(a) = Df(g(a)) \circ Dg(a)$ matrix mult = composition

$$\mathbb{R}^n \xrightarrow{g} \mathbb{R}^m \xrightarrow{f} \mathbb{R}^p$$

Idea of proof: we go to the definition of the derivative and show that

$$\lim_{x \to a} \frac{\|(f\circ g)(x) - (f\circ g)(a) - Df(g(a)) \circ Dg(a)(x-a)\|}{\|x-a\|} \to 0$$

$$\frac{\| fg(x) - fg(a) - Df(g(a)) \circ Dg(a)(x-a) \|}{\|x-a\|}$$

$$= \frac{\| fg(x) - fg(a) - Df(g(a))(g(x)-g(a)) + Df(g(a))(g(x) - g(a) - Dg(a)(x-a) \|}{\| x-a \|}$$

$$\leq \frac{\|fg(x) - fg(a) - Df(g(a)(g(x)-g(a))\|}{\|x-a\|} + \frac{\| Df(g(a))(g(x)-g(a)) - Dg(a)(x-a)\|}{\|x-a\|}$$

etc.

Other properties of the derivative

They may all be proved by going to the definition of the derivative and showing that the given expression for the derivative satisfies the definition.

Theorem 10 (i) and (ii)
Let f, g : R^n -> R^m be differentiable and let c, d be scalars.
Then cf + dg is differentiable and
D(cf + dg) (a) = c Df (a) + d Dg (a)

D is linear.

Example: $D(2x+3x^2) = 2D(x) + 3D(x^2)$

Parts (iii) and (if) of Theorem 10 do not have much use because they only apply to differentiable functions f,g : R^n -> R

(iii) The product gf is differentiable and
D(fg) (a) = g(a) Df(a) + Dg(a) f(a)

(iv) If g is never zero then f/g is differentiable and
D(f/g) (a) = g(a) Df(a) - f(a) Dg(a) / [g(a)]^2

Exercise 1. If f : R^n -> R is differentiable, prove that x -> f^2(x) + 2f(x) is differentiable and find its derivative in terms of Df .

Most questions get you to calculate D(f ∘ g) in particular cases, and they can mostly be done by first computing f ∘ g and then finding the derivative without the chain rule.

Example

f(u,v) = (u-v, 2v^2) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

g(x,y,z) = (xy+z, x^2 + y^2 + z^2) $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

Find D(f ∘ g) (1,0,1)

Example (like Ex 33):

Let $c = c(t) : R \to R^2$ have $c(1) = (3,4)$ and $c'(1) = (1,2)$.
Let $f = f(x,y) : R^2 \to R$ have $\partial f/\partial x\,(3,4) = -1$.
Suppose that $d/dt\,(f \circ c) = 5t$.
Find $\partial f/\partial y\,(3,4)$.

Solution. We use $\frac{df}{dt}(1) = \frac{\partial f}{\partial x}(c(1))\frac{dx}{dt}(1) + \frac{\partial f}{\partial y}c(1)\,\frac{dy}{dt}(1)$

$$= \frac{\partial f}{\partial x}(3,4)\frac{dx}{dt}(1) + \frac{\partial f}{\partial y}(3,4)\frac{dy}{dt}(1)$$

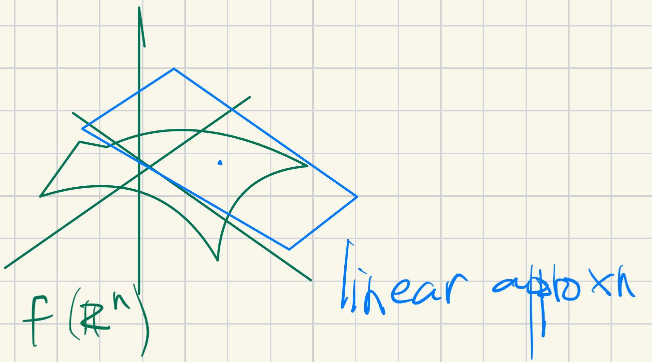
$$= (-1)\cdot 1 \quad + \frac{\partial f}{\partial y}(3,4)\cdot 2$$

$$= 5t = 5$$

$$-1 + 2\frac{\partial f}{\partial y}(3,4) = 5, \quad \frac{\partial f}{\partial y}(3,4) = 3$$

Why might we expect the chain rule to be true?

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$



Same for g.

$D(g \circ f)$ gives the best linear approxn.

To get the best approximation to $g \circ f$ we compose the best approximation for f with the best approximation for g.